The Fundamentals of

PHYSICS

Course I - Mechanics

Math Supplement

Robert A. Cohen
Physics Department
East Stroudsburg University

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Introduction

This supplement is not designed to teach you the math. Rather, the assumption is that you have already learned the math and either have forgotten some aspects or may not have fully understood some aspects at the time. This is a *supplement*, after all, and the purpose is to provide support for the mathematical concepts utilized to do physics, not teach you the mathematical concepts.

Consequently, you will find that topics are covered only briefly, with the main points being emphasized rather than practice with all of the finer points that you might encounter. If you understand the main points, you should be able to comprehend and carry out most of the physics discussed in the textbook, although you will still need practice working out the details of some mathematical steps.

If you find you are confused by anything in this supplement, you are encouraged to consult the instructor or a mathematics textbook.

Contents

1. Mathematical Notation

1.1 Symbols

1.1.1 Variables

Algebra[i] is a technique that uses symbols to represent values. These symbols are typically Roman letters in italic font (like a, b, c, etc.). For example, we could say that we have three values, a, b and c, that are related (for some reason) as follows:

$$a \text{ plus } b \text{ equals } c.$$

✍ We use the *italic* font for these letters.

✍ In physics, we also tend to us a lot of Greek letters as in:

$$\rho \text{ equals } m \text{ divided by } V,$$

where "ρ" is the Greek letter "rho".

We use the word **variable** to describe the value represented by the letter-symbol.

The word "variable" implies that the value can vary and, in a sense, this is true since, technically, a letter-symbol can represent any value. However, in some cases, there is a single, particular value that the letter represents and either we don't know what it is or, if we know what it is, it is easier to write the letter than the value.

For example, in physics we find that a particular combination of variables always equals the value 6.67×10^{-11} Nm^2/kg^2. This is quite cumbersome to write over and over again, so we frequently just substitute the letter G to represent this value. We typically call such values **constants** instead of *variables* but algebraically they are treated the same way.

[i] See chapter 2 for algebraic techniques.

✍ Although each variable is represented by a single letter, there are many cases when a single letter can be ambiguous. In those cases, it is popular to add a description in the form of a subscript. For example, the letter F may be used to not only indicate "force" but also "net force" or "initial force" or "the force exerted by object A" and so on. In the expression

$$F_{\mathrm{net}} = F_1 + F_2.$$

the subscript "net" is used to indicate that the F on the left-hand side actually represents the value of the *net* force. The numbers used for the subscripts on the right-hand side are used to distinguish between the two other forces.

✍ Sometimes, the subscript is an abbreviation. For example, F_{i} might represent the *initial* value of the force (where "i" stands for "initial")

✍ Usually the subscripts will be in roman font rather than italic font but this is not a universal rule.

1.1.2 Actions

We also use symbols to represent "actions", like addition (+) and equivalence (=). Although we could use any symbol for the actions and any symbol for the values, in practice we tend to use letters for the values and non-letters for the actions, just to keep things straight.

So, the two expressions above would be written as follows:

$$\begin{aligned} a + b &= c \\ \rho &= m/V \end{aligned}$$

✍ For multiplication, various symbols have been used. For example, the symbol $\times$ is popular in grade school and for scientific notation (see chapter 4). Since the $\times$ symbol can often be confused with the letter x, which would represent a variable abbreviation, people sometimes used a dot instead. For example, $m \cdot s$ would represent the product of m times s.

Often, no symbol is used at all is used for multiplication. For example, in the expression

$$F = ma$$

the two values represented by the m and a are multiplied together. The assumption, then, is that each single letter represents a separate value and so two letters together represent the product of two values.

There are a couple of exceptions to the "rule" that actions are represented by symbols. In some cases, an "action" will be represented by a letter. For example, in

$$\Sigma F = ma$$

the Greek letter "Σ" (upper-case sigma) represents the action of adding up all of the F's (i.e., a sum). Or, someone might write

$$f(x) = x^2$$

where the letter f is supposed to represent whatever action (on x) is described on the right-hand side of the expression. If $f(x) = x^2$ then this is supposed to mean that the letter f represents the action of squaring whatever is inside the parentheses (which immediately follow it). For example, if $f(x) = x^2$ then $f(3)$ would equal 9 (i.e., the square of 3).

1.2 Units

Values typically include both a number and a unit. For example, if you went to the store to buy two loaves of bread, the value "two loaves of bread" contains a number (2) and a unit (loaves of bread). Each letter-symbol in an equation typically has both a number and a unit.

When doing actions on values, the action acts on both the number and the unit. For example, if we multiply a by b, where a and b are variable abbreviations, we get ab. If a happens to be (2 m) and b happens to be (3 s), where "m" and "s" are unit abbreviations, then ab equals (6 m·s). Notice that the units are multiplied together just as the numbers are multiplied together.

2. Algebra

2.1 Simplifying equations

Algebraic equations are used to describe the relationship between variables (where each variable is represented by a letter). Depending on the situation, however, you may want to *simplify* the equation.

There are lots of rules for how one goes about simplifying equations. The most important rules are as follows:

Rule #1: Treat variables abbreviations like unit abbreviations.

For example, we know that two apples plus three apples equals five apples. Mathematically, we could use "a" to represent the unit "apple". Then, the relationship could be written as "2a + 3a = 5a."

In a similar way, $2x + 3x = 5x$, where x is a variable abbreviation.

On the other hand, we cannot add two apples plus three bananas and get five apple-bananas or some such sort of thing. And, similarly, $2x + 3y \neq 5xy$.

So, if you want to add or subtract variables, make sure they are the same.

> ✍ Just like units, variables can be multiplied together, squared, cubed, etc. Just like you can't add x and y to get something like xy, you can't add x and x^2 to get something like x^3.

Rule #2: Consider how the equation might be simplified if actual numbers were used instead of letters.

For example, x^2 multiplied by x^3 equals x^5. It does not equal x^6. One can tell it is not x^6 simply by plugging in some value for x, calculating x^2 and x^3, multiplying them together, and comparing the answer with x^6.

As another example, consider multiplying $(3a + b)$ by two. The product equals $(6a + 2b)$. It does not equal $(6a + b)$. Again, one can tell by plugging in some values for a and b.

Checking each step of your work by plugging in numbers may seem cumbersome but after a while you will identify the pattern and then you can apply the pattern without checking. That is, after all, the purpose of algebra.

2.2 Rearranging equations

Suppose you have an equation as follows:

$$2(3a + b) = 2a - b$$

Notice how the variable a appears twice, as does the variable b. Sometimes it is easier to use an equation that doesn't have individual variables appearing more than once. Consequently, it helps to know how to "rearrange" an equation.

Again, there are lots of rules about rearranging equations. The most important rules are as follows:

Rule #1: You can do anything you want to the expression as long as you do the *same thing to both sides.*

For example, if you are given

$$a + b = c$$

you can subtract b from the left side only if you also subtract b from the right side. Doing so, gives

$$a + b - b = c - b$$

which simplifies to

$$a = c - b.$$

The expression "$a = c - b$" represents the same relationship as the expression "$a + b = c$".

Rule #2: Keep your eye on the prize.

In other words, what you do to each side depends on what you want to end up with. In this case, if you want an expression that has "$a =$" then each step you make must get you closer to that form. There are lots of manipulations you can do to an equation. You must consider which manipulations will get you closer to the prize.

Consider the expression given at the beginning of this section:

$$2(3a + b) = 2a - b.$$

In accordance with rule #1 we could add 2 to both sides. However, that would only give us

$$2(3a + b) + 2 = 2a - b + 2$$

which isn't any simpler than what we had before.

To illustrate what I mean by rule #2, suppose we want to end up with an expression that had a by itself on the left-hand side such that the equation had the form "$a =$". That means we want to get rid of the other stuff on the left-hand side.

This "other stuff" includes the number 2, the number 3 and the variable b. If you think about the order in which the calculations are made, starting with a, we find that first we multiply a by 3, then add b, then multiply the whole thing by two. When simplifying the equation, we tend to work backward. First we'll divide the whole thing by two, then subtract b then divide by 3.

Step-by-step, this becomes:

1. First divide both sides by 2 to get

$$\begin{aligned} 2(3a + b) &= 2a - b \\ 2(3a + b)/2 &= (2a - b)/2 \\ (3a + b) &= (a - b/2) \end{aligned}$$

2. Then subtract b from both sides to get

$$\begin{aligned} (3a + b) &= (a - b/2) \\ (3a + b) - b &= (a - b/2) - b \\ 3a &= a - (3b/2) \end{aligned}$$

3. Then divide both sides by 3 to get

$$\begin{aligned} 3a &= a - (3b/2) \\ 3a/3 &= [a - (3b/2)]/3 \\ a &= (a/3) - (b/2) \end{aligned}$$

Notice that I've made some simplifications to the right-hand side along the way. If you don't follow the simplifications I've made, remember to check them by plugging in some values for a and b. Eventually, you'll find the pattern and can just use the pattern.

2.3 Combining equations

Many times it is necessary to combine two or more equations to obtain a new equation.

The act of combining equations is not difficult. What is difficult is knowing how to combine the equations to get where you want to go.

For example, suppose you had the following two equations:

$$\begin{aligned} a &= 2b + 3 \\ b &= 2a - 4 \end{aligned}$$

and you want an expression that only has a in it (no b's).

One way to do this is to recognize that the second expression tells you the value of b in terms of a. Thus, you can replace every b with $(2a - 4)$.

Consequently, if we replace the b in the first expression with $(2a - 4)$, we get

$$\begin{aligned} a &= 2b + 3 \\ &= 2(2a - 4) + 3. \end{aligned}$$

The expression now has a's only. We can now simplify, if we want, to get

$$\begin{aligned} a &= 2(2a - 4) + 3 \\ a &= (4a - 8) + 3 \\ a &= 4a - 5 \\ a - 4a &= 4a - 5 - 4a \\ -3a &= -5 \\ -3a(-1/3) &= -5(-1/3) \\ a &= 5/3. \end{aligned}$$

The most difficult step is the first step. There are many first steps that you can take but only some will take you closer to the goal. That is why it is important to know where you are going.

In a sense, the process is similar to those puzzles where you are given a word like "FACE" and are asked to change one letter at a time, getting a new word each time, and eventually get a totally different word, like "MILK".

For example, if we start with "FACE" then we can replace "F" with "P" to get "PACE" and then replace the "C" with an "L" to get "PALE" and so on as follows: FACE → PACE → PALE → PILE → MILE → MILK.

Notice that the changes each time are not arbitrary. Each change is made deliberately to get us closer to the word we are after.

2.4 Simultaneous equations

If we have a single equation with one unknown (say, x), there should be a solution that we can find using algebraic techniques. In other words, if we have a single relationship with several variables and all of the variables are known except for one, we can solve for the unknown variable.

For example, the equation "$2x+3y=4$" has two variables, x and y. Suppose we know that $y=1$. That means that y is known. The only unknown variable, then, is x. Since we have only one unknown variable, we can algebra to solve for the unknown variable. In this case, x is found to be $^1/_2$.

Will we always have only one unknown variable?

No. In fact, the focus of this chapter is on situations where there is more than one unknown variable.

What do we do when there is more than one unknown variable?

When you have more than one unknown variable, you can still solve for the unknown variables but only if you have the same number of equations as you have unknown variables.[i]

[i]The equations also have to be independent. In other words, one of the equations cannot simply be a rearrangement of another equation or a combination of two or more of the other equations.

For example, suppose you had the following two equations:

$$\begin{aligned} x + y &= 3 \\ y &= 1 \end{aligned}$$

and you wanted to solve for x. Only one of the equations has x in it but that equation also has y, which is also "unknown" and so a second equation is necessary. In this case, the second equation only has one unknown (y) and so we can solve for that unknown and use that in the first equation.

I call this the "sequential" approach to solving for multiple unknowns. The variable you want is only present in combination with another unknowns. However, the other unknowns can be solved *first* via other equations and *then* we can use those values to solve for the one we want.

A more complicated situation is when the two variables are "intertwined" so one has to use both equations to solve for either unknown.

For example, suppose you had the following two equations:

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \end{aligned}$$

Since these are two independent equations, you can solve for the x and y values but you must combine the two equations to get rid of one of the unknowns.

HOW?

There are several ways. Here is one way. First rearrange one of the equations to be in the form "$x =$" or "$y =$". For example, the first equation can be written as "$x = 3 - y$".

THEN WHAT? WE CAN'T GET x IF WE DON'T KNOW y.

Since we know x is equivalent to $(3 - y)$, we can replace every instance of x in the second equation with $(3 - y)$. The second equation becomes

$$(3 - y) - y = 1.$$

This is now a single equation with a single unknown, y. We can use algebra to solve for the unknown. Try it. You should find that y equals 1.

WHAT IS THE VALUE OF THE OTHER VARIABLE, x?

Once you know one of the variables (in this case, y), you can use that value in one of the original equations to find the value of the other variable. For example, if we replace y by 1 in the first equation, we get

$$x + (1) = 3$$

and we can solve this to get that x is equal to 2.

Check Point 2.1: Use the following two relationships

$$\begin{aligned} x + 2y &= 4 \\ 2x - y &= 9 \end{aligned}$$

to obtain the values of x and y.

2.5 Derivation of Quadratic Equation

Quadratic equations are equations that include the square of the unknown variable (see section 5.2).

For a quadratic equation of the form

$$ax^2 + bx + c = 0$$

there are up to two possible values of x that solve the equation. These two values are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To derive the quadratic formula, we first assume the solution has the following form:

$$x = \alpha \pm \beta$$

Then, knowing that a quadratic can then be rewritten as

$$(x - \alpha - \beta)(x - \alpha + \beta) = 0,$$

we expand to get

$$x^2 - (\alpha + \beta)x - (\alpha - \beta)x + (\alpha + \beta)(\alpha - \beta) = 0$$

which, after simplifying, becomes

$$x^2 - 2\alpha x + (\alpha^2 - \beta^2) = 0.$$

We then equate this with $x^2 + (b/a)x + (c/a) = 0$, the original equation (divided through by a), to get that

$$\begin{aligned} -2\alpha &= (b/a) \\ (\alpha^2 - \beta^2) &= (c/a) \end{aligned}$$

and, solving both equations for α and β, get

$$\begin{aligned} \alpha &= -b/(2a) \\ \beta &= \sqrt{b^2 - 4ac}/(2a) \end{aligned}$$

giving us the quadratic formula.

Check Point 2.2: Use the quadratic formula to obtain the two values of x that solve the following equation: $x^2 + 10x = -3$.

Answer to checkpoints

2.1 $x = 4.4$, $y = -0.2$

2.2 -0.31 and -9.7

3. Units

3.1 International System of Units (SI)

The scientific community uses the International System of Units, sometimes referred to as SI or the metric system. In this system, there are only a couple of basic physical quantities and each one is assigned a unique unit. For our purposes, there are four basic physical quantities: length, mass, time and temperature. In SI, the associated units are meters, kilograms, seconds and Kelvin.

Abbreviation	**Name**	**Quantity**
kg	kilograms	mass
m	meters	length
s	seconds	time
K	Kelvin	temperature

Units for all other quantities are derived in terms of these basic units (for this reason, this is sometimes called the MKS system – M for meters, K for kilogram and S for seconds).

Abbreviation	**Name**	**Quantity**
°C	degrees Celsius (K - 273.15)	temperature
Hz	Hertz (cycles/s)	frequency
J	Joules (kg·m^2·s^{-2})	energy
N	Newtons (kg·m·s^{-2})	force
Pa	Pascals (N/m^2)	pressure
rad	radians (m/m)	angle
W	Watts (J/s)	power

Prefixes are then used to represent factors of ten for each unit (see table 3.1). For example, the "kilo" represents 10^3. Consequently, a kilometer is equivalent to a thousand meters and a kilogram is equivalent to a thousand grams.

Abbreviation	**Name**	**Quantity**
Y	Yotta-	10^{24}
Z	Zetta-	10^{21}
E	Exa-	10^{18}
P	Peta-	10^{15}
T	Tera-	10^{12}
G	Giga-	10^{9}
M	Mega-	10^{6}
k	kilo-	10^{3}
h	hecto-	10^{2}
da	deka-	10^{1}
d	deci-	10^{-1}
c	centi-	10^{-2}
m	milli-	10^{-3}
μ	micro-	10^{-6}
n	nano-	10^{-9}
p	pico-	10^{-12}
f	femto-	10^{-15}
a	atto-	10^{-18}
z	zepto-	10^{-21}
y	yocto-	10^{-24}

Table 3.1: SI prefixes (source: http://physics.nist.gov/cuu/Units/prefixes.html).

✍ You don't need to memorize them (after all, there is always the table) but you should become familiar with the most commonly used prefixes, which are Mega-, kilo-, centi-, milli- and micro-.

✍ The abbreviation for "micro" is the only one listed that is a Greek letter. It is the Greek letter "mu" (some people mistakenly think it is the Roman letter u).

✍ All of the abbreviations are one letter except for *deca*-.

✍ Many people get centi and milli confused. If it helps, consider that the "centi-" prefix comes from the same root as "century" (100 years) and the "milli-" prefix comes from the same root as "mile" (1000 paces) and "millenium" (1000 years).

3.2 Non-SI units

There are many units that not part of SI but are still popular. The following can be used with the metric prefixes.

Abbreviation	**Name**	**Quantity**	Value (in SI units)
$^\circ$	degree	angle	$\pi/180$ rad
bar	bar	pressure	10^5 Pa
h	hour	time	60 min (3600 s)
L	liter	volume	1000 cm^3
min	minutes	time	60 s
ua	astronomical unit	length	$\sim 1.496 \times 10^{11}$ m

Others are outside SI and are not used with the metric prefixes.

Abbreviation	**Name**	**Quantity**	Value (in SI units)
ft	feet	length	12 in (0.3048 m)
in	inches	length	(2.54 cm)
mi	miles	length	5,280 ft (1609.344 m)

3.3 Unit conversion

After all of the work involved in getting an answer to problems, one's work is not necessarily finished. This is because the answer needs to be interpreted and, in some cases, revised so that the audience can interpret the answer appropriately.

For example, suppose you multiply (5 ft/s) by 10 h. The result is 5 ft·h/s. This is a distance but it isn't in units that anyone is familiar with. Consequently, you cannot leave the answer as "5 ft·h/s." You must convert it to a unit that is familiar.

The problem is that the answer contains two different units of time (i.e., hours and seconds). To simplify, convert one of the time units so that the two units are the same.

This can be done by simply replacing one of the units with its equivalent. For example, since 1 h = 3600 s, replace "h" with "3600 s" to get

$$5\ \mathrm{ft}\frac{3600\ \mathrm{s}}{\mathrm{s}}$$

which simplifies to 18,000 ft.

Sometimes the result is still hard to interpret because the number is too large or too small. For example, is the distance from class to where you live larger than 12,350,000 inches? Most people cannot tell without converting the number to a more reasonable unit.

Check Point 3.1: Are you taller than 0.001 miles?

In this case, since

$$1\ \mathrm{mi} = 5,280\ \mathrm{ft}$$

we could convert 18,000 ft to miles by replacing "ft" with "(1/5280) mi". This gives

$$\frac{18,000}{5,280}\ \mathrm{mi}$$

or 3.41 mi.

Even this may not be satisfactory, since it is not in SI. To convert it to SI, replace "mi" with "1609.344 m" to get

$$3.41(1609.344\ \mathrm{m})$$

or 5490 m.

This should then be converted to a more reasonable number by using the metric prefixes (i.e., 5.49 km).

Check Point 3.2: The density of water is about 1 g/ml (or 1000 kg/m^3).
(a) Estimate the size of a drop of water.
(b) A molecule of water consists of one oxygen atom and two hydrogen atoms. From (a), estimate the number of water molecules in a drop of water.
(c) Estimate the size of the ocean.
(d) From (a) and (c), estimate the number of drops in the ocean.
(e) Which is larger: the number of water molecules in a drop of water or the number of water drops in the ocean?

Answers to checkpoints

3.1 0.001 miles is equivalent to 5 feet, 3.36 inches.

3.2 (a) A typical drop is about 0.05 ml (20 drops per milliliter), any reasonable estimate is fine,
(b) The molecular masses of oxygen and hydrogen are about 16 g/mol and 1 g/mol, respectively, and so one mole of water molecules has a mass of about 18 g; from the density of water, a drop of water has a mass of abuot 0.05 g; dividing that by 18 g/mol gives 0.002778 moles and since there are 6.022×10^{23} molecules in a mole (Avogadro's number), this means that there are (0.002778 mol) $\times$ (6.022×10^{23} molecules/mol) or $1.67 \times 10^{2}1$ water molecules in a drop of water,
(c) The surface area of the Earth is $4\pi r^2$, where r is equal to 6.37×10^6 m; about 75% of the Earth is covered by oceans, so multiply the area by 0.75 to get the surface area of the oceans; the average depth of the ocean is about 5000 m, so multiply the surface area by 5000 m; this gives 2.55×10^{18} m^3,
(d) Divide the ocean volume (2.55×10^{18} m^3) by the volume of a water drop (0.05 ml/drop) to get 5.1×10^{19} drop $\cdot$ m^3/ml; a milliliter is equivalent to a cubic centimeter (cm^3) while a cubic meter is equivalent to a million cubic centimeter (m^3 is equal to

$(100\ \text{cm})^3$); making that conversion, we get that m^3/ml is equal to 10^6 and so there are 5.1×10^{25} drops in the ocean,
(e) There are more drops in the ocean than molecules in a drop of water

4. Scientific Notation

One way of getting rid of big or small numbers is to change the units (see chapter 3).

Another way of getting rid of big or small numbers is to use scientific notation. In scientific notation, the factor of ten is written explicitly rather than in a metric prefix. For example, instead of writing 5490 m as 5.49 km, we could instead write it as 5.49×10^3 m.

Such notation also makes it much easier to multiply or divide very big and very small numbers. For example, what is 12,350,000,000,000 s times 0.0000000000350 s?

Not only does it take a long time to write down all of the zeroes associated with very big or very small numbers, but it is hard to readily see how many zeroes are present without carefully counting them. These two problems are addressed via a technique called **scientific notation**.

Scientific notation essentially takes all of the zeroes and wraps them up in the form of 10^n, where n is some integer. For example, since ten million (7 zeros) is the same as $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$ (ten multiplied by itself 7 times), we can write ten million as 1×10^7. Since a number like 55,000,000 is just 5.5 times ten million, we can write 55,000,000 as 5.5×10^7.

Likewise, a number can have lots of zeros if it is really small, e.g., 0.0000001. Since one-ten-millionth (6 zeros) is the same as 1/10/10/10/10/10/10/10 (one divided by ten 7 times), we can write one-ten-millionth as 1×10^{-7}. Since a number like 0.00000055 is just 5.5 times one-ten-millionth, we can write 0.00000055 as 5.5×10^{-7}.

It is easier to multiply and divide very large or small numbers when they written in scientific notation because we can quickly identify how many zeroes are present. For example, suppose you had to multiply 5.5×10^7 by 5×10^2. We can simply perform this multiplication in two steps. First, we multiply 5.5 by 5 to get 27.5. Then, we multiply 10^7 by 10^2 to get 10^9. The final result is 27.5×10^9 or 2.75×10^{10}.

✍ By convention, the first number is written from 1 to 9. Consequently, it is better to write the answer as 2.75×10^{10} (instead of 27.5×10^{9}). For example, a number like 12,300 would be written as 1.23×10^{4} rather than 123×10^{2}.

5. Equation Types

If someone gives you an equation relating two variables, you should be able to identify certain characteristics of the relationship simply by recognizing certain aspects of the equation.

To help you with this interpretation, I will classify equations into four types. There are many more than these four types but these are four types that you will run in most frequently and consequently you should be familiar with their characteristics.

The four types are:

- **Linear**: Linear relationships have the form $y = ax + b$ where x and y are the variables and a and b are constants (numbers). A special case of a linear relationship is when b is zero. In that case, y and x are **directly proportional** to each other. Examples of linear relationships include
  - The relationship betwen the circumference of a circle C and the circle's diameter D
    $$C = \pi D$$
    where π is the constant of proportionality (it is a number that, when rounded, is 3.1415927).
  - The relationship between the mass m of a material and its volume V
    $$m = \rho V$$
    where ρ is the constant of proportionality (depends on the material) and is known as the *density*.
  - The relationship between the current I passing through a piece of material and the electrical energy (per charge) V that is converted to heat or light
    $$V = IR$$

where R is the constant of proportionality (depends on the size, type and shape of the material) and is known as the *resistance*. This is known as *Ohm's law*.

- The relationship between the force F exerted on a spring and the distance Δs that the spring stretches beyond its original length (i.e., the length when no force was exerted on it)

$$F = k(\Delta s)$$

where k is the constant of proportionality (depends on the spring) and is known as the *spring constant*. This is known as *Hooke's law*.

- **Quadratic**: Quadratic relationships have the form $y = ax^2 + bx + c$ where x and y are the variables and a, b and c are constants (numbers). A special case of a quadratic relationship is when b and c are zero. In that case, y is directly proportional to the square of x. Examples of quadratic relationships include

  - The relationship between the surface area A of a ball and the ball's diameter D

$$A = \pi D^2$$

where π is the constant of proportionality (it is a number that, when rounded, is 3.1415927).

  - The relationship between the height h that a ball is dropped from rest and the time t it takes for the ball to hit the ground

$$h = kt^2$$

where k is the constant of proportionality and is equal to 4.9 m/s^2.

- **Inverse**: Inverse relationships have the form $xy = a$ where x and y are the variables and a is a constant (number). Examples of inverse relationships include

  - The relationship between the pressure P exerted on or by a container of gas and its volume V (assuming constant temperature)

$$PV = k$$

where k is the constant of proportionality (depends on the gas and the temperature). This is known as *Boyle's law*.

- **Independent**: Independent relationships have the form $y = a$ where y is a variable and a is a constant (number). Examples of independent relationships include
  - The relationship between the the time t it takes for a ball to hit the ground (dropped from rest at a given height) and its mass m

    $$t = k$$

    where k is the constant of proportionality and depends on the height the ball is dropped.
  - The relationship between the period T of a pendulum and the its amplitude θ (the maximum angle it makes with the vertical as it swings back and forth)

    $$T = k$$

    where k is the constant of proportionality (depends on the length of the pendulum).
  - The relationship between the force F needed to hold a syringe plunger steady when there is a vacuum inside the syringe and the distance s that the plunger is pulled out (assuming the plunger is still inside the syringe)

    $$F = k$$

    where k is the constant of proportionality (depends on the atmospheric pressure and the cross-sectional area of the plunger).

For each type, I'll identify its properties. That way, if you encounter an equation of one of these types, you'll be able to identify its properties, regardless of the variables that are involved.

5.1 Linear relationships

5.1.1 Zero intercepts

The simplest kind of relationship is called a *linear* relationship. An example of a linear relationship is the relationship between the circumference of a circle C and the circle's diameter D. The relationship is

$$C = \pi D \tag{5.1}$$

where π is the constant of proportionality (it is a number that, when rounded, is 3.1415927).

To examine the properties of such an expression, let's plug in various values of D and find the associated circumferences. Or, alternately, we could take some circles (cans provide good examples of circles) and measure their diameters and circumferences.

Suppose I measure a circle of diameter 2.0 cm. According to the expression above, the circumference of such a circle would be about 6.3 cm.

If we then measure a circle of diameter 4.0 cm (i.e., twice the radius), then the equation says we will get a circumference of about 12.6 cm.

Notice that the circumference doubles when the diameter doubles. This represents a situation where the two variables (diameter and circumference) are *directly proportional* to each other.

When two variables are directly proportional to each other, their ratio always has the same value. So, if one increases, the other must increase so that the ratio has the same value.

Thus, if one variable doubles the other must double also. If one variable triples, the other must triple also. This pattern applies for any number. For example, if one variable is multiplied by a number n, the other variable must also increase by a factor n (see footnote[i]).

It turns out that all relationships between two variables that are directly proportional to each other can be written as $y/x = a$ (or $y = ax$) where x and y are the two variables and a is some known number (*pi* in our case).

---

[i] Algebraically, this can be shown by first showing that, since the ratio must always be the same, the ratio of the old values must equal the ratio of the new values. Mathematically, if the two variables are x and y, we can represent their ratios as follows:

$$\frac{y_{\text{new}}}{x_{\text{new}}} = \frac{y_{\text{old}}}{x_{\text{old}}}.$$

Then, if x_{new} is n times greater than x_{old}, we can replace x_{new} by nx_{old}:

$$\frac{y_{\text{new}}}{nx_{\text{old}}} = \frac{y_{\text{old}}}{x_{\text{old}}}.$$

Multiplying both sides by nx_{old} and simplifying, we get

$$y_{\text{new}} = ny_{\text{old}}.$$

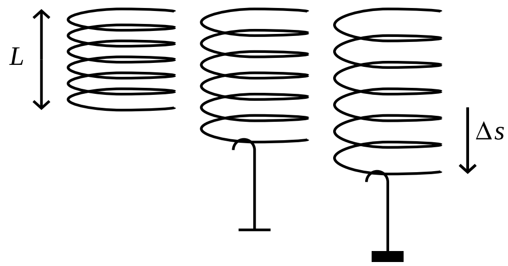

Figure 5.1: A hanging spring. When weights are hung from the spring, the spring stretches.

Does the constant π have a unit?

No. This is because the two variables, diameter and circumference, have the same unit. Technically, the unit could be cm/cm but since cm/cm is equivalent to one, we don't need to write the units.

Check Point 5.1: I have two cans. Can A has a diameter that is ten times bigger than can B's diameter. What is the ratio of their circumferences?

5.1.2 Non-zero intercepts

When the diameter of a circle is equal to zero, so is its circumference. This is a special case of a linear relationship, called a direct proportion.

This is not the case for all linear relationships. Consider, for example, the force needed to stretch a spring. In figure 5.1, there is a hanging spring. Without any weight hanging on it, the spring has a length L. As we add more weight (force pulling it down), the spring stretches The variable Δs indicates how far the spring stretches (beyond the original length L).

The relationship between force F exerted on the spring and the distance Δs that the spring stretches beyond its initial length is

$$\Delta s = \alpha F \tag{5.2}$$

where the variable α represents the constant of proportionality between the force and the distance.

This is a direct proportion relationship because when no weights are pulling down on the spring, the "stretch" is zero (i.e., the spring length is L).

However, suppose instead we wanted the relationship between the force F and the total length of the spring, which I'll indicate as ℓ. The total length ℓ is the original length L plus how far the spring stretches Δs.

Suppose, for instance, that the initial length of the spring was 10 cm. When the force is zero, ℓ is equal to 10 cm.

✍ The relationship between F and Δs is usually written in the form $F = k\Delta s$, where the constant of proportionality k is just the inverse of α and is known as the *spring constant*. This form is known as *Hooke's law*, named after the English scientist Robert Hooke born in 1635 who, among other things[ii] first identified this relationship.[iii]

Check Point 5.2: If we double the strength of the force F pulling on the spring, does the length ℓ of the spring also double?

5.2 Quadratic relationships

An example of a quadratic relationship is the relationship between the surface area A of a sphere and the sphere's diameter D. The relationship is

$$A = \pi D^2 \tag{5.3}$$

where π is the same number represented by π in the relationship for a circle's circumference (equation 5.1).

To examine the properties of such an expression, let's plug in various values of D and find the associated surface areas. Or, alternately, we could take some spheres (balls provide good examples of spheres) and measure their diameters and surface areas. Suppose I measure some spheres and get the

[ii]Robert Hooke also wrote the first book describing observations made through a microscope and was the first person to use the word "cell" to identify microscopic structures.

[iii]This particular relationship does not hold for all springs. In addition, it won't hold if the spring is stretched too much. I've been treating it as true for the purposes of illustrating a linear relationship.

following values:[iv]

Diameter	Surface Area
2.1 cm	14 cm^2
2.7 cm	23 cm^2
3.4 cm	36 cm^2
3.7 cm	43 cm^2
5.4 cm	92 cm^2

Check Point 5.3: What would the surface area be for a ball whose diameter is 4.5 cm?

The general form of a quadratic relationship between variables x and y is

$$y = ax^2 + bx + c.$$

In the case of a sphere's surface area, x and y are diameter and surface area, respectively. The constants b and c are zero and the constant a is π.

When b and c are zero, y is directly proportional to the *square* of x. In the relationship for the surface of a sphere, this means that if the diameter doubles, the surface area increases by a factor equal to the square of two (i.e., 4). If the diameter triples, the surface area increases by a factor equal to the square of three (i.e., 9). This pattern applies for any number. For example, if the diameter is multiplied by a number n, the surface area must increase by a factor of n^2 (see footnote[v]).

[iv] The abbreviation "cm^2" stands for "square centimeters".

[v] Algebraically, this can be shown by first showing that, since $y = ax^2$ then the ratio y/x^2 must be constant:

$$\frac{y_{\text{new}}}{x_{\text{new}}^2} = \frac{y_{\text{old}}}{x_{\text{old}}^2}.$$

Then, if x_{new} is n times greater than x_{old}, we can replace x_{new} by nx_{old}:

$$\frac{y_{\text{new}}}{(nx_{\text{old}})^2} = \frac{y_{\text{old}}}{x_{\text{old}}^2}.$$

Multiplying both sides by $n^2 x_{\text{old}}^2$ and simplifying, we get

$$y_{\text{new}} = n^2 y_{\text{old}}.$$

Conversely, if the surface area doubles, the diameter must increase by a factor equal to the square root of two (i.e., about 1.41). If the surface area triples, the diameter must increase by a factor equal to the square root of three (i.e., about 1.73). Again, this pattern applies for any number. For example, if the surface area is multiplied by a number n, the diameter must increase by a factor of $\sqrt{n}$ (see footnote[vi]).

Check Point 5.4: I have two balls. Ball A has a diameter that is ten times bigger than ball B's diameter. What is the ratio of their surface areas?

5.3 Inverse relationships

In the situations observed so far, if one variable increases, so does the other. In an inverse relationship, the opposite happens: if one variable increases, the other decreases.

An example of an inverse relationship is the relationship between the volume V of an enclosed gas and its pressure P (assuming the temperature is held constant):

$$P = k\frac{1}{V}$$

or

$$PV = k \tag{5.4}$$

[vi] Algebraically, this can be shown in the same way as with the previous footnote except that instead of replacing x_{new} by nx_{old}, we replace y_{new} by ny_{old}. As before, we start by recognizing that ratio of y to x^2 is constant:

$$\frac{y_{\text{new}}}{x^2_{\text{new}}} = \frac{y_{\text{old}}}{x^2_{\text{old}}}.$$

Then, if y_{new} is n times greater than y_{old}, we can replace y_{new} by ny_{old}:

$$\frac{ny_{\text{old}}}{x^2_{\text{new}}} = \frac{y_{\text{old}}}{x^2_{\text{old}}}.$$

Inverting both sides and multiplying both sides by ny_{old} and simplifying, we get

$$x_{\text{new}} = \sqrt{n}x_{\text{old}}.$$

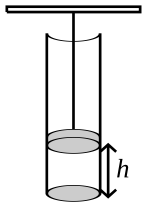

Figure 5.2: A syringe made up of a tube and a plunger. A platform is connected to the plunger such that weights can be placed upon the platform, pushing the plunger down into the syringe tube.

where k is the constant of proportionality.[vii]

To examine the properties of such an expression, let's plug in various values of V and find the associated pressures. Or, alternately, we could take a syringe, made up of a tube and a plunger (see figure 5.2), cover the bottom end (so no air can escape through that end) and lay weights on the plunger (to increase the pressure) so that the plunger goes down (into the syringe tube), decreasing the volume. For each pressure, we measure the volume taken up by the air inside the syringe. Suppose I make some measurements and get the following values:

Pressure	**Volume**
1.01 atm	5.00 cm^3
1.14 atm	4.44 cm^3
1.27 atm	3.99 cm^3
1.78 atm	2.84 cm^3
2.42 atm	2.09 cm^3

This would be what one would get by using equation 5.4 with k equal to 5.06 atm·cm^3.

WHAT IS AN atm?

It is an abbreviation for a unit called an atmosphere. An atmosphere is a unit of pressure. Notice that this particular unit is abbreviated using three letters instead of just one.

[vii]This relationship is called Boyle's law, in honor of Robert Boyle, who lived from 1627 to 1691, an Irish scientist who studied the properties of air and was one of the first to identify the relationship between the pressure and volume of a gas. A French scientist named Edme Mariotte (1620-1684) also identified the relationship at about the same time so the relationship is sometimes called Mariotte's law or the Boyle-Mariotte law.

How do we know that "atm" is an abbreviation for a single unit rather than three separate units ("a", "t" and "m") multiplied together?

If I am multiplying units together, I will include a space or a dot between them (e.g. "a t m" or "a·t·m").

Check Point 5.5: Based on the data provided in this section, what would be the volume of the air inside the syringe when the pressure is 2.00 atm?

When two variables are inversely proportional, the product is constant. In the relationship for the syringe, this means that if the pressure doubles, the volume is halved. If the pressure triples, the volume is cut to one-third of its value. In general, if the pressure is multiplied by a number n, the volume must decrease by a factor of $1/n$ (see footnote[viii]).

Check Point 5.6: I have a syringe. If I double the pressure acting on the plunger, what happens to the volume of the air inside the syringe?

5.4 Independent relationships

An independent relationship is one in which one variable does not depend at all on the other. In other words, you can change one of the variables all you want and the other variable will not change.

[viii] Algebraically, this can be shown by first showing that, since $xy = a$ then

$$y_{\text{new}} x_{\text{new}} = y_{\text{old}} x_{\text{old}}.$$

Then, if x_{new} is n times greater than x_{old}, we can replace x_{new} by nx_{old}:

$$y_{\text{new}}(nx_{\text{old}}) = y_{\text{old}} x_{\text{old}}.$$

Dividing through by nx_{old} and simplifying, we get

$$y_{\text{new}} = y_{\text{old}}/n.$$

For example, a pendulum can be constructed by attaching a ball to a string and hanging the string from a rod or post. If the ball is pulled to one side and released, it swings back and forth. The period of a pendulum is the time it takes to swing back and forth and return to its release point. The amplitude is the distance it swings back and forth. We can change the amplitude by simply releasing the ball from various positions.

We find the period is the same, regardless of the amplitude.[ix]

How is an independent relationship expressed mathematically?

In this situation, since the time is always equal to the same number, we might just write

$$T = k$$

where T is the period and k is the value.

The problem with this is that the period may depend on other things, like the length of the pendulum. In other words, it isn't independent of *everything*. The equation does not say what it is independent of. So, it might be clearer to add a qualification, like

$$T = k \text{ for any amplitude.}$$

Check Point 5.7: A pendulum is found to have a period of 0.23 seconds. What do you expect the period to be if you decrease the amplitude to half of what it was?

5.5 Other relationships

Most of the relationships we'll encounter will be either linear, inverse, quadratic or independent. However, there will be a few that don't fit into any of these categories.

For example, the farther a planet is from the sun, the longer it takes to go around the sun. The relationship between the time it takes and the radius

[ix]Scientists usually honor Galileo as the "discoverer" of this relationship.

of the orbit is

$$\frac{T^2}{R^3} = 1\ \mathrm{yr}^2/\mathrm{ua}^3$$

where T is the period and R is the radius of the planet's orbit around the sun.

WHAT DOES "yr" AND "ua" STAND FOR?

They are abbreviations for the units "year" and "astronomical unit" (the distance from the sun to the earth).

Although this relationship looks different from the other relationships we've examined, it can still be expressed as a ratio that is equal to a constant. Consequently, if you know the period, you can solve for the radius of orbit. Conversely, if you know the radius, you can solve for the period.

WHAT HAPPENS TO THE PERIOD IF YOU DOUBLE THE RADIUS?

To answer that question, it is easier to first solve the expression for T. Multiply both sides by R^3 and then take the square root of both sides to get

$$T = \sqrt{(1\ \mathrm{yr}^2/\mathrm{ua}^3)R^3}$$

Since we can write the square root of something as the number raised to power of one-half, let's rewrite this expression as

$$\begin{aligned} T &= \left[(1\ \mathrm{yr}^2/\mathrm{ua}^3)R^3\right]^{1/2} \\ &= (1\ \mathrm{yr}^2/\mathrm{ua}^3)^{1/2}(R^3)^{1/2} \\ &= (1\ \mathrm{yr}^2/\mathrm{ua}^3)^{1/2}R^{3/2} \end{aligned}$$

You should now be able to see that if you increase the radius by a factor of n, the period will increase by a factor of $n^{3/2}$.

WHAT DOES IT MEAN TO RAISE SOMETHING TO THE POWER OF THREE-HALVES?

Raising something to the power of three-halves is the same as first cubing the number and then taking the square-root of the number (or, conversely, first taking the square-root of the number and then cubing it).

So, for example, suppose we triple the radius. Cube that to get a factor increase of 27 and then take the square root of that to get a factor increase of 5.2. This means that the period will increase by a factor of 5.2 when the radius is tripled.

✍ It isn't necessary to first solve the expression for T. We could, instead, do it in steps. First, if R triples, then R^3 must increase by a factor of 3^3 or 27. Next, if R^3 increases by a factor of 27 then T^2 must likewise increase by a factor of 27 (since the ratio T^2/R^3 is constant). Finally, since T^2 increases by a factor of 27, T must increase by a factor equal to the square root of 27 (i.e., 5.2).

Check Point 5.8: If the period is tripled, what happens to the radius?

Practice with scaling

Problem 5.1: If you take an ideal gas and let it expand or compress in an insulated environment, the gas will cool or heat adiabatically such that its temperature and pressure are related as follows:

$$\frac{T^7}{P^2} = k$$

where T is the temperature, P is the pressure and k is a constant. What happens to the temperature if the pressure is doubled?[x]

Problem 5.2: If you hang some weight from a spring and then pull down and release, the weight will bob up and down on the spring. Suppose the period of the bobbing is related to the mass of the weight as follows:

$$T^2 m = k$$

where T is the period, m is the mass of the weight and k is a constant. If I double the mass hanging from the spring, what happens to the period of the bobbing?

Problem 5.3: Some materials, like gold, have a high density. Consequently, a small volume can be very heavy. When comparing the masses of various

[x]This relationship, known as Poisson's equation, assumes the temperature is measured in an absolute scale like Kelvin or Rankin.

materials, I find that I only need a small volume of more-dense materials (like gold) to balance the mass of less-dense materials (like wood). The relationship between the mass and the volume is

$$m = \rho V$$

where ρ is the constant of proportionality (depends on the material) and is known as the density*. If I double the mass, what happens to the volume?*

Problem 5.4: If we take a pendulum, measure its length and period, then change its length, re-measure the period, and so on, we get the following measurements:

L	*T*
30 cm	*1.1 s*
50 cm	*1.4 s*
100 cm	*2.0 s*

where L is the length and T is the period (using T instead of P because the period represents a time).
(a) What is the relationship between the length and the period?
(b) Use your relationship to predict the period when the length is 80 centimeters.

Answers to checkpoints

5.1 10

5.2 Yes

5.3 63.6 cm^2

5.4 100

5.5 2.53 cm^3

5.6 It gets halved

5.7 0.23 s (remain the same)

5.8 It doubles (actually it goes up by a factor of 2.08)

6. Graphing

6.1 Linear relationships

6.1.1 Zero intercepts

As mentioned in chapter 5, the simplest kind of relationship is called a *linear* relationship. An example of a linear relationship is the relationship between the circumference of a circle C and the circle's diameter D (equation 5.1):

$$C = \pi D$$

where π is the constant of proportionality.

To clarify the relationship, the values are graphed in figure 6.1. Each point on the graph represents a particular value of C and a particular value of D. The D value is given by the horizontal axis (indicated by the "Diameter (cm)") and the C value is given by the vertical axis (indicated by the "Circumference (cm)").

Important: For the discussion that follows, we will assume that the graph is constructed such that the increments along each axis is *uniform*. In other words, a difference of 1 cm is represented by the same distance along the axis, regardless of where you are along the axis.

The first thing you probably notice is that the points lie along a straight line. This characteristic is more obvious when a straight line is drawn through the points (see right graph in figure 6.1). This is the most obvious characteristic of linear relationships (i.e., when you plot values that satisify the relationship, you get a straight line).

The line in figure 6.1 goes through the origin (0,0). Will a linear relationship always go through the origin?

No. A linear relationship is graphed as a straight line and a straight line could be drawn in lots of ways that do not go through the origin (0,0).

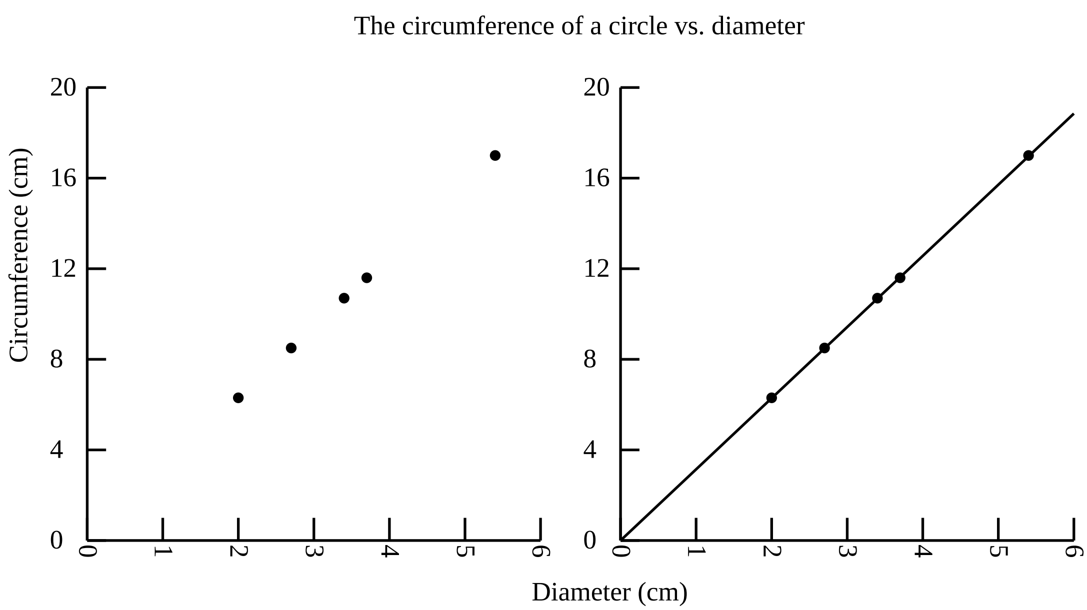

Figure 6.1: [left] A graph showing the circumference of a circle for certain diameters. [right] As in the left figure but with a straight line drawn through the points.

Then why does it go through the origin in this case?

The line goes through the origin *in this case* because a circle of zero diameter must have zero circumference.

Linear relationships that include the point (0,0) are a special type of linear relationship called the *direct proportion*. In such relationships, not only are both variables zero at the same time but the two variables are directly proportional to each other (as mentioned in chapter 5.

Note that the number π not only represents the proportional constant and the ratio of the two variables, but it also represents the slope of the line drawn on the graph. The slope of a "flat" line is zero, whereas a line inclined up the page has a positive slope and a line inclined down the page has a negative slope.

The slope between any two points is defined as the ratio $\Delta y/\Delta x$, where Δx is the difference in the x values of the two points and Δy is the difference in the y values of the two points.

For a linear relationship, such as the one shown in figure 6.1, the line is straight and such has the same slope everywhere. As can be shown mathematically, the slope has the same value as the proportionality constant.[i] The convention in mathematics is to represent the slope by the letter m. Consequently, we could write the general form of a linear relationship as $y = mx + b$.

Check Point 6.1: What is the slope of the line drawn in figure 6.1?

---

[i]If $y = ax + b$, where a and b are constants, then

$$\begin{aligned}
\Delta y/\Delta x &= (y_2 - y_1)/(x_2 - x_1) \\
&= ([ax_2 + b] - [ax_1 + b])/(x_2 - x_1) \\
&= (ax_2 + b - ax_1 - b)/(x_2 - x_1) \\
&= (ax_2 - ax_1)/(x_2 - x_1) \\
&= a(x_2 - x_1)/(x_2 - x_1) \\
&= a.
\end{aligned}$$

6.1.2 Non-zero intercepts

When the diameter of a circle is equal to zero, so is its circumference. As mentioned before, this is a special case of a linear relationship, called a direct proportion, and the graph shows a straight line that intersects the origin (0,0).

The intersection of the line with the y axis is called the *y-intercept*. For the case shown in figure 6.1, the y-intercept is zero.

This is not the case for all linear relationships. Consider, for example, the force needed to stretch a spring. As mentioned in chapter 5, the relationship between force F exerted on the spring and the distance Δs that the spring stretches beyond its initial length is given by equation 5.2

$$\Delta s = \alpha F$$

where the variable α represents the constant of proportionality between the force and the distance.

Let's suppose, for example, that α equals 2 cm/N. That means that graphing Δs vs. F results in a straight line with slope equal to 2 cm/N (see left side of figure 6.2).

This is a direct proportion relationship because when no weights are pulling down on the spring, the "stretch" is zero (i.e., the spring length is L).

However, suppose instead we wanted the relationship between the force F and the total length of the spring, which I'll indicate as ℓ. The total length ℓ is the original length L plus how far the spring stretches Δs.

Suppose, for instance, that the initial length of the spring was 10 cm. When the force is zero, ℓ is equal to 10 cm. When we plot ℓ vs. F (instead of Δs vs. F), it looks like the line has been shifted upward (see right side of figure 6.2). This is because the values have all been increased by 10 cm.

The slope is still α but now there is a y-intercept equal to L. Mathematically, the relationship is

$$\begin{aligned} \ell &= \Delta s + L \\ &= \alpha F + L \end{aligned}$$

The relationship between F and ℓ is not a direct proportion.

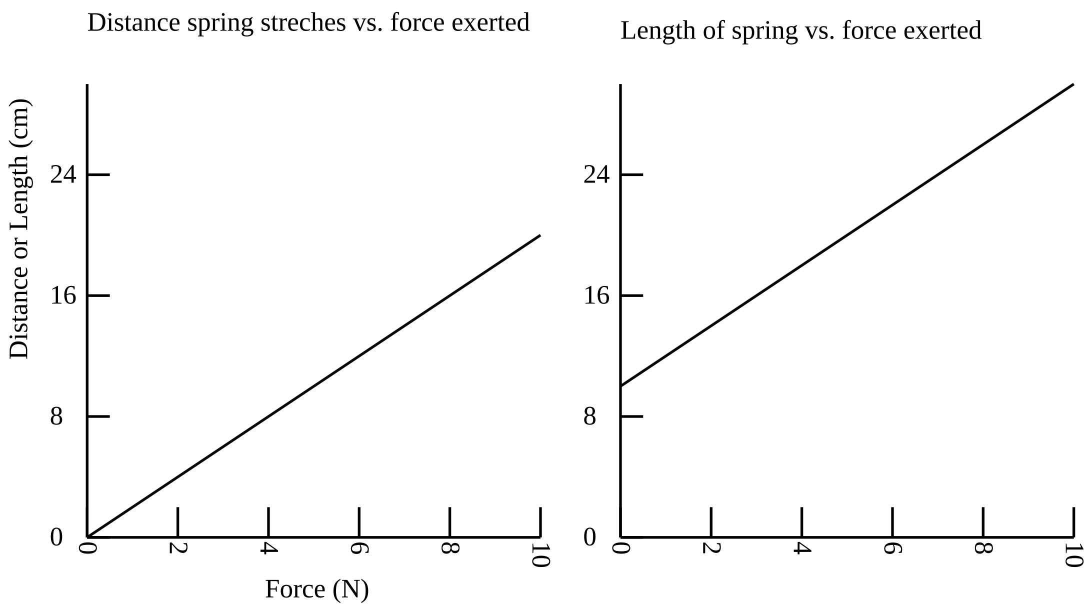

Figure 6.2: [left] A graph showing how far a spring stretches (beyond its initial length) as a function of the force exerted on the spring. [right] A graph showing the total length of a spring as a function of the force exerted on the spring.

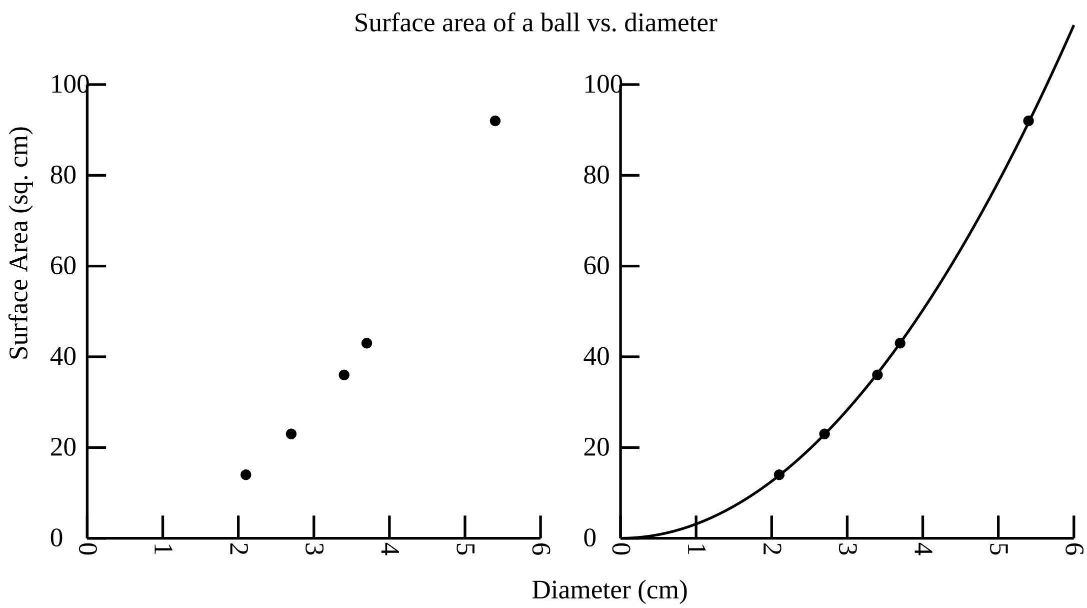

Figure 6.3: [left] A graph showing the surface area of a sphere for certain diameters. [right] As in the left graph but with a curve drawn through the points as specified by equation 5.3.

Check Point 6.2: (a) What is the slope of the line in figure 6.2?
(b) What would be the slope of the line if ℓ vs. F was plotted instead of F vs. ℓ?

6.2 Quadratic relationships

An example of a quadratic relationship is the relationship between the surface area A of a sphere and the sphere's diameter D (equation 5.3):

$$A = \pi D^2$$

Again, we can gain some insight into this relationship by graphing the data (see figure 6.3).

At first glance, you might think the points lie along a straight line. However, upon closer examination, it is clear that there is a slight curve to it. Quadratic relationships, unlike linear relationships, will display a curve. The curve, however, might be slight and so, depending on the range of data you are observing, may be difficult to see.

When the curve described by equation 5.3 is drawn (see right side of figure 6.3), the curvature is more apparent.

The line in figure 6.3 goes through the origin (0,0). Will a quadratic relationship always go through the origin?

No. The curve goes through the origin *in this case* because a sphere of zero diameter must have zero surface area.

In section 5.1.1, the number π represented the slope of the line. Is that also the case here?

No. The slope of a curved line varies. Initially (with small diameters), the slope is not much greater than one. As the diameter gets larger, the slope becomes greater.

How do determine the slope of the curve?

As before, the slope between any two points is defined as the ratio $\Delta y/\Delta x$, where Δx is the difference in the x values of the two points and Δy is the difference in the y values of the two points.

For a quadratic relationship, such as the one shown in figure 6.3, taking two points on the line will give the slope of the straight line connecting those two points. Since the line is curved, that slope will not equal the slope at either of the two points chosen but rather a point somewhere in between.

To get a slope, you must either take two points that are very close to the location on the line that you wish to compute the slope[ii] As can be shown mathematically[iii], the slope does not have the value of π but rather $2\pi D$. As D increases, so does the slope.

[ii]The actual definition is to take two points that are infinitesimally close together. If the difference in y is dy and the difference in x is dx, then the slope would be defined as the ratio dy/dx.

[iii]Mathematically, if $y = ax^2$ then the slope between any two points is:

$$\begin{aligned}
\Delta y/\Delta x &= (y_2 - y_1)/(x_2 - x_1) \\
&= (ax_2^2 - ax_1^2)/(x_2 - x_1) \\
&= a(x_2^2 - x_1^2)/(x_2 - x_1) \\
&= a(x_2 - x_1)(x_2 + x_1)/(x_2 - x_1)
\end{aligned}$$

Check Point 6.3: What is the slope of the line drawn in figure 6.3 when the diameter is 4 cm?

6.3 Inverse relationships

An example of an inverse relationship is the relationship between the volume V of an enclosed gas and its pressure P (assuming the temperature is held constant; see equation 5.4):

$$PV = k$$

Again, we can gain some insight into this relationship by graphing the data (see figure 6.4).

Again, at first glance, you might think the points lie along a straight line. However, upon closer examination, it is clear that there is a slight curve to it, much like the curve seen in the quadratic relationship between the diameter of a sphere and the surface area. However, unlike that relationship, the curve is going downward, not upward.

When the curve described by equation 5.4 is drawn (see right side of figure 6.4), the curvature is more apparent.

WHY DOESN'T THE CURVE IN FIGURE 6.4 GO THROUGH THE ORIGIN (0,0)?

Because, according to equation 5.4, if the presure was zero, the air in the syringe would expand unimpeded to an infinite size. It would not contract to zero volume.

Check Point 6.4: (a) Based on the relationship shown in figure 6.4, what should the volume be when the pressure is zero?
(b) What should the volume be when the pressure is infinite?

$$= \quad a(x_2 + x_1).$$

If x_1 and x_2 are very close to one another in value, then their sum is twice the value of x_1 (or twice the value of x_2; it wouldn't matter since they have essentially the same value). Thus, the slope has a value equal to $a2x_1$.

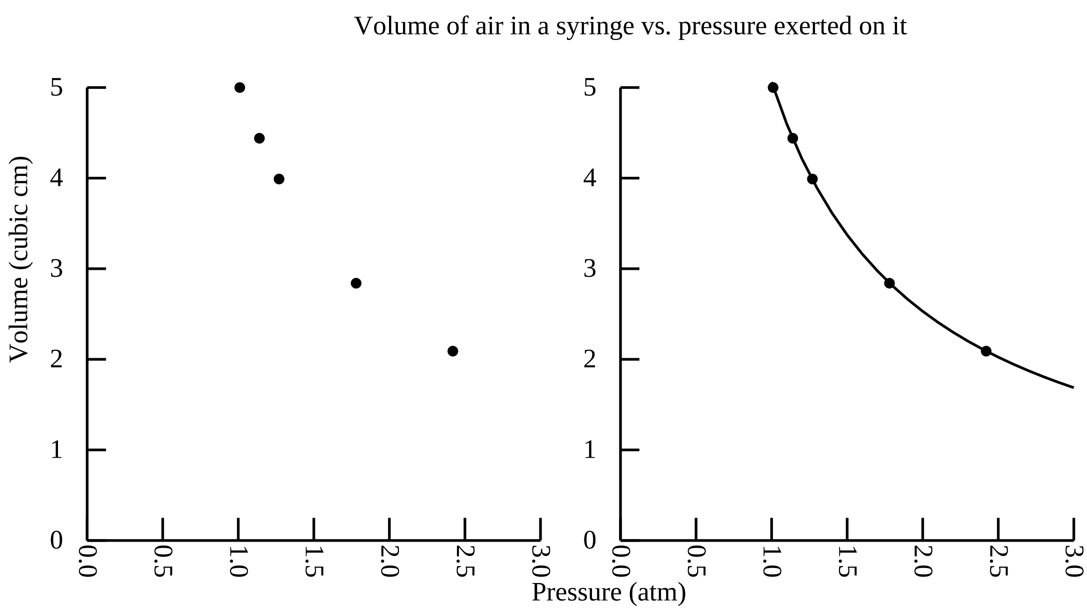

Figure 6.4: [left] A graph showing the volume of air inside a syringe for certain pressures. [right] As in figure 6.4 but with a curve drawn through the points as specified by equation 5.4.

Answers to checkpoints

6.1 The slope is around 3 cm/cm (the actual value is *pi*, which is about 3.1415927).

6.2 (a) 2 cm/N, (b) 0.5 N/cm

6.3 About 25 cm^2/cm (or 25 cm)

6.4 (a) Infinite, (b) zero

7. Averages

7.1 Meaning of average

To illustrate what is meant by an average, consider the situation where you take four exams, receiving grades of 70, 80, 90 and 100. What is your average grade?

You can probably guess that the average is somewhere in the middle. In this case, the average is 85, which is exactly in the middle.

How did you figure out the average?

In general, one figures out the average by adding up all of the values and dividing by the number of values:

$$X_{\text{avg}} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} \tag{7.1}$$

In this case, the sum of all the values is 340. There are four values, so we divide 340 by four to get 85.

Can we simply add the max and min and take the average?

In this case, that would work (i.e., add 70 and 100 and divide by two). However, that won't always work. Consider, for example, the following four grades: 70, 100, 100 and 100. How do you figure out the average now?

You should be able to see that the average should be higher than before since there are more 100's. If we simply took the average of the maximum (100) and minimum (70) values, we'd get the same average as before (85). That wouldn't be right.

So how do we obtain the average?

Use equation 7.1. First, add up all of the values to get 370. Then, divide by the number of values (4) to get 92.5.

Notice that the average is larger than before.

CAN THE AVERAGE EVER BE MORE THAN THE MAXIMUM OR LESS THAN THE MINIMUM?

No.

Example 7.1: Suppose you have 16 values. Ten of them equal 2000 each. The other six equal −1000 each. What is the average value?

Answer 7.1: Using equation 7.1, we add up all of the values and divide by the total number of values. Since we have 10 values of 2000 each and six equal to −1000 each, we have

$$\begin{aligned} X_{\text{avg}} &= (X_1 + X_2 + X_3 + \cdots + X_n)/n \\ &= \Big((2000 \times 10) + (-1000 \times 6)\Big)/16 \\ &= (20000 - 6000)/16 \end{aligned}$$

which gives an average value of 875.

WHY DID YOU MULTIPLY 2000 BY TEN?

Rather than write out "$2000 + 2000 + 2000 + \cdot$" and so on for a total of ten values of 2000 each, I simply multiplied 2000 by ten. I did the same with the −1000 values. Since I had six of those, I simply multiplied −1000 by six.

Check Point 7.1: Suppose you have 40 values. Fifteen of them equal 2 each. Five more equal zero each. The remaining 20 equal -1 each. What is the average value?

7.2 Continuous data

So far we have only examined situations where we are given discrete (separate) values and asked to determine the average of the separate values. A more common situation in physics is one where we have a value that varies continuously over some period of time.

Example 7.2: A particular force is 2000 N for 10 seconds and then −1000 N for 6 seconds. What is the average value of the force?

Answer 7.2: We can find the average force using the same technique discussed above. Basically, we can use equation 7.1 to add up all of the values and divide by the total number of values. Since the force is 2000 N for ten seconds and then −1000 N for six seconds, we have

$$\begin{aligned} F_{\text{avg}} &= (F_1 + F_2 + F_3 + \cdots + F_n)/n \\ &= \big((2000 \text{ N}) \times (10 \text{ s}) + (-1000 \text{ N}) \times (6 \text{ s})\big)/(16 \text{ s}) \\ &= \big((20000 \text{ N} \cdot \text{s}) - (6000 \text{ N} \cdot \text{s})\big)/(16 \text{ s}) \\ &= (14000 \text{ N} \cdot \text{s})/(16 \text{ s}) \end{aligned}$$

which gives an average value of 875 N.

Notice that the procedure is the same as before but since we are dealing with continuous values we multiply by the period of time. We are basically *weighting* each force by the period of time it is acting. As such, when dealing with continuous values, it is more proper to rewrite equation 7.1 as follows:

$$X_{\text{avg}} = \frac{X_1 \Delta t_1 + X_2 \Delta t_2 + X_3 \Delta t_3 + \cdots + X_n \Delta t_n}{\Delta t_{\text{total}}} \tag{7.2}$$

Check Point 7.2: A car experiences an acceleration of 2 m/s^2 for 15 seconds and then 0 m/s^2 for 5 seconds more and finally -1 m/s^2 for 20 more seconds. What is the average acceleration during the 40 seconds.

7.3 Graphing techniques

When dealing with continuous variables, we find that graphs can help identify the average. For example, consider the graph in figure 7.1.

In this figure, the net force (indicated by the solid line) is shown to be 2000 N for the first eight seconds and −1000 N for the last eight seconds. In this case, the average value is midway between the maximum (2000 N) and minimum (−1000) values and is indicated by the dashed line (at 500 N).

In comparison, consider the graph in figure 7.2. In that case, the net force (indicated by the solid line) is shown to be 2000 N for the first ten seconds,

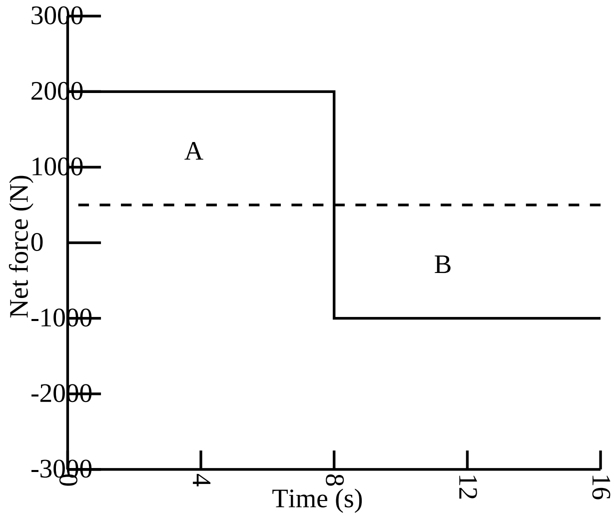

Figure 7.1: A graph showing the net force varying over time (solid line) with average value indicated by dashed line.

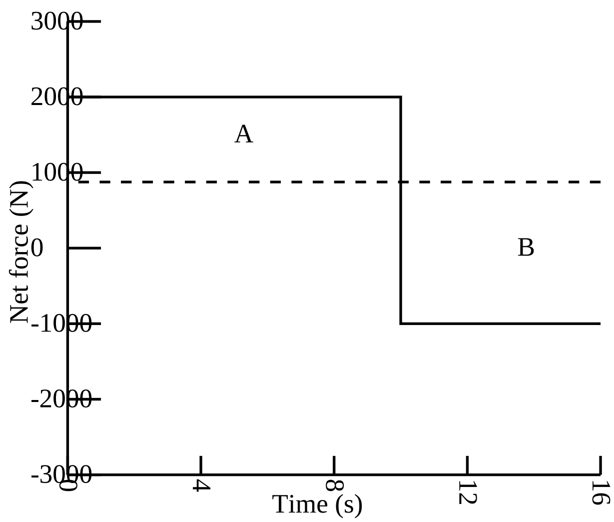

Figure 7.2: A graph showing the net force varying over time (solid line) with average value indicated by dashed line.

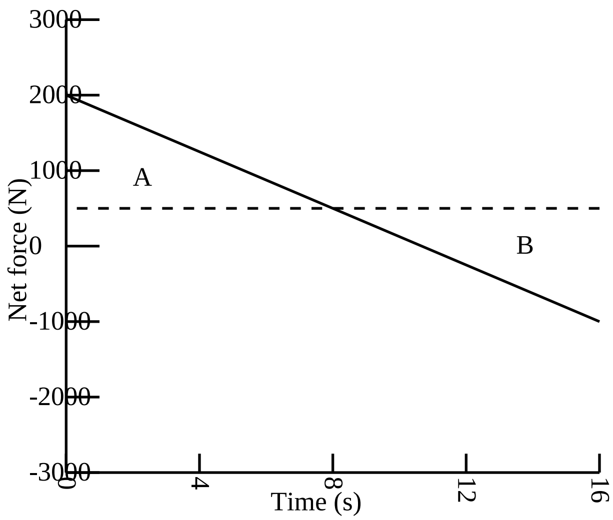

Figure 7.3: A graph showing the net force varying over time (solid line) with average value indicated by dashed line.

a longer portion of the time than in figure 7.1. Consequently, the average value is shifted higher (indicated by the dashed line at 875 N).

WHAT DO THE LETTERS A AND B REPRESENT IN FIGURE 7.1 AND 7.2?

They are drawn to indicate one of the properties of the average (indicated by the dashed line in the figures). It turns out that the area of the box above the line (indicated by the letter A) is equal to the area below the line (indicated by the letter B).

WHY IS THAT IMPORTANT?

Because it allows us to estimate the average from the graph without having to go through equation 7.2.

To illustrate the benefit of that, consider the graph in figure 7.3.

In this case, the net force varies linearly from 2000 N at the start to -1000 N at the end. The average is still midway between the maximum and minimum (indicated by the dashed line at 500 N). Rather than "boxes" above and below the dashed line, we have triangles. Still, the area of the triangles are the same, refective of the dashed line being the average.

Example 7.3: In figure 7.4, which of the dashed lines represents the average

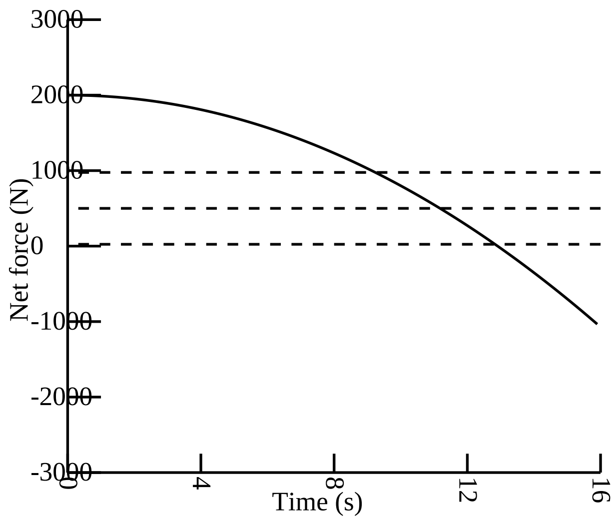

Figure 7.4: A graph showing the net force varying over time (solid line) with three possible average values indicated by the dashed lines.

value of the net force?

Answer 7.3: The top one. That is the only one where the "area" bounded by the solid line above and below the dashed line are equal.

Check Point 7.3: In figure 7.5, which of the dashed lines represents the average value of the net force? Explain your choice.

7.4 Linear relationships

When a variable varies linearly[i], a plot of the value results in a straight line (constant slope), as in figure 7.3. In this special case (as in the case with figure 7.3), the average value lies midway between the maximum and minimum values.

$$X_{\text{avg}} = \frac{X_i + X_f}{2} \tag{7.3}$$

[i]See section 5.1 for information about linear relationships.

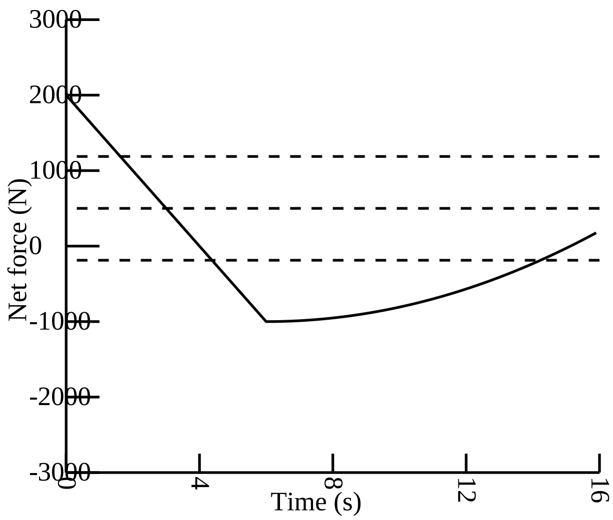

Figure 7.5: A graph showing the net force varying over time (solid line) with three possible average values indicated by the dashed lines.

For variables that vary linearly with time, the average value is midway between the maximum and minimum values.

Check Point 7.4: Suppose an object experiences an acceleration proportional to time in the following way: $a = (-1.0\ \text{m/s}^3)t$.
(a) What is the initial acceleration ($t = 0$) *and final acceleration* ($t = 3$ s) *over a time interval of three seconds?*
(b) Calculate the average acceleration during the time interval from zero to three seconds.

Answers to checkpoints

7.1 0.25

7.2 0.25 m/s^2

7.3 Bottom line

7.4 (a) zero and -3.0 m/s^2, (b) -1.5 m/s^2

8. Trigonometry

8.1 Sine

In figure 8.1, a circle is drawn with three thin lines extending outward from the center of the circle. These three lines are labeled A, B and C.

Since a circle has a constant radius, the length of each line is the same. The orientations are different, however. Line A is drawn horizontally. Line B is drawn diagonally. And line C is drawn vertically.

The angle between C and A is $90°$. The angle between B and A is something less than $90°$. Well represent the angle with the Greek letter θ (theta). Consequently, the angle between C and B is $(90° - \theta)$.

In this section, well focus on the thick line labeled Y. This line extends from line A to point P, which is where line B intersects the circle.

What do you know about this line?

For one, it represents how high point P is above line A. In addition, the length of this line varies depending upon the angle θ.

When θ is zero, what is the length of Y?

The length of Y is also zero. If θ is zero, then B lies on top of A.

When θ is $90°$, what is the length of Y?

The length of Y would then be the same length as the radius of the circle. If θ is $90°$, then B lies on top of C.

When θ is $45°$, what is the length of Y?

One might think the answer would be half the radius. This is incorrect. If you draw the situation, youll find that Y is actually a little bit bigger than half the radius. It turns out that Y is about 70% the size of the radius (or $0.7R$, where R is the radius).

For what value of θ is the length of Y equal to one-half of the radius?

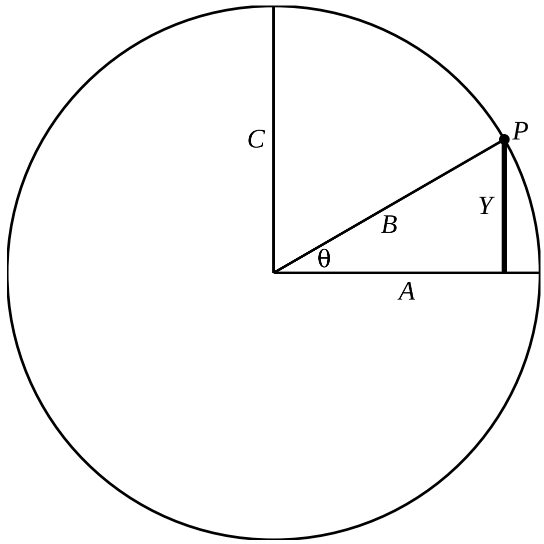

Figure 8.1: A circle with several segments, discussed in the text.

From the previous answer, it should be clear that θ must be less than 45°. However, it probably isnt clear exactly what the angle is. It turns out that the angle is 30°.

Given a value of θ (like 30°), how does one figure out the length of Y?

There is no straightforward way. Fortunately, other people have determined what the ratio of Y to R (radius) would be for any value of θ. The ratio is called the **sine**.

The values can be determined with the help of a calculator. For example, if one wants to determine the sine when the angle is 30°, one could type 30 on the calculator and then press the "sin" button. This should give an answer of 0.5. If it does not, check that the calculator is interpreting the 30 as 30 degrees and not some other unit.

Given a value of Y (like 0.5), how does one figure out the angle θ?

Again, the best way is to use the calculator. The procedure for obtaining the angle given the length is the opposite of what one does if one has the angle and wants to determine the length. For example, if one wants to determine the angle when the sine is 0.5, one could type 0.5 on the calculator and then

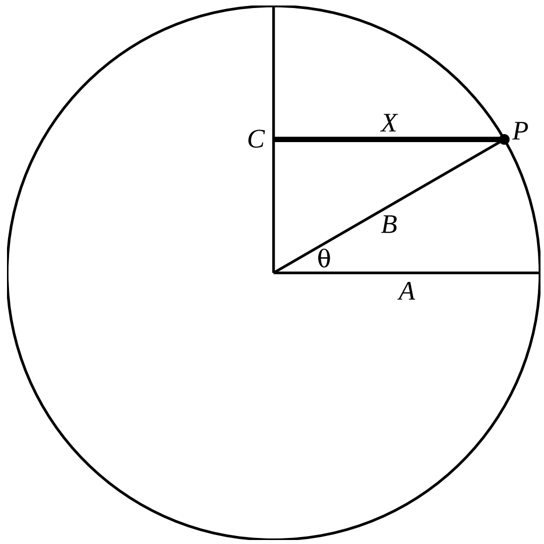

Figure 8.2: A circle with several segments, discussed in the text.

press the "$\sin^{-1}$" button (one some calculators, this is done by first pressing a "INV" button and then the "sin" button). This should give an answer of 30. If it does not, check that the calculator is interpreting the 30 as 30 degrees and not some other unit.

When I ask the calculator for the sine of an angle greater than 180°, it gives me a negative answer. What gives?

In the figure on the previous page, we have defined "upward" as positive. If θ is greater than 180°, then B is oriented downward. This corresponds to a negative Y.

8.2 Cosine

Figure 8.2 is similar to figure 8.1. The difference is that, instead of a line Y it has a line X. This line extends from line C to point P, which is where line B intersects the circle. As with Y, the length of this line varies depending upon the angle θ.

What do you know about this line?

For one, it represents how far point P is from line C. In addition, the length of this line varies depending upon the angle θ.

When θ is zero, what is the length of X?

The length of X is the same length as the radius. If θ is $0°$, then B lies on top of A.

When θ is $90°$, what is the length of X?

The length of X would then be zero. If θ is $90°$, then B lies on top of C.

When θ is $45°$, what is the length of X?

As before, the answer is not half the radius. If you draw the situation, youll find that X is actually a little bit bigger than half the radius. It turns out that X is about 70% the size of the radius (or $0.7R$). Both X and Y are the same when θ is $45°$.

For what value of θ is the length of X equal to one-half of the radius?

It should be clear that θ must be greater than $45°$. However, it probably isnt clear exactly what the angle is. It turns out that the angle is $60°$.

Given a value of θ (like $30°$), how does one figure out the length of X?

As for Y, there is no straightforward way. Fortunately, other people have determined what the ratio of X to R would be for any value of θ. The ratio is called the **cosine**.

The values can be determined with the help of a calculator. For example, if one wants to determine the cosine when the angle is $30°$, one could type 30 on the calculator and then press the "cos" button. This should give an answer of about 0.866. If it does not, check that the calculator is interpreting the 30 as 30 degrees and not some other unit.

Given a value of Y (like 0.5), how does one figure out the angle θ?

As with the sine, the procedure for obtaining the angle given the length is the opposite of what one does if one has the angle and wants to determine the length. For example, if one wants to determine the angle when the cosine is 0.5, one could type 0.5 on the calculator and then press the "$\cos^{-1}$" button (one some calculators, this is done by first pressing a "INV" button and then the "cos" button). This should give an answer of 60. If it does not, check that the calculator is interpreting the 30 as 30 degrees and not some other unit.

When I ask the calculator for the cosine of an angle greater than 90°, it gives me a negative answer. What gives?

In the figure, we have defined "to the right" as positive. If θ is greater than 90°, then B is oriented toward the left. This corresponds to a negative X.

8.3 Cosine and sine as perpendicular components of a vector

In the two diagrams, we can interpret Y as the component of B in the direction of the vertical axis (typically called the y axis). Similarly, we can interpret X as the component of B in the direction of the horizontal axis (typically called the x axis).

From the discussion in the previous sections, it was found that the ratio of Y to B is called the sine and that ratio is only dependent upon the angle θ. As long as the angle θ is known, the ratio of Y to B is known. In other words, the sine only depends on the angle θ. Since the sine only depends upon the angle, we typically write the sine as $\sin(\theta)$, indicating that it is a function of θ only. Similarly, we write the cosine as $\cos(\theta)$.

Since the length of B is R (radius), we can also write that $\sin(\theta) = Y/R$ and $\cos(\theta) = X/R$. Similarly, we can write

$$\begin{aligned} Y &= R\sin(\theta) \\ X &= R\cos(\theta). \end{aligned}$$

In other words, the vertical component is obtained via the sine. The horizontal component is obtained by the cosine.

Of course, how one defines vertical and horizontal is arbitrary. It all depends on the direction we call 0°. In the diagrams, 0°pointed horizontally to the right. It doesnt have to. Since the cosine isnt always horizontal, it is safer to state the relationship as follows:

- The cosine provides the component in the same direction as that indicated by 0°
- The sine provides the component perpendicular to the direction indicated by 0°

9. Powers of Ten

9.1 Positive integers

You should already be familiar with a number like 10^2 ("ten squared"). This is the same as 10×10 ("ten times ten").

Example 9.1: What is 10^3? **Answer 9.1:** 1000.

It is pretty straightforward to recognize that, for each power of ten, another zero is added to the number. Ten squared (10^2) is one hundred (2 zeroes) and ten cubed (10^3) is one thousand (3 zeroes).

Example 9.2: What is 10^1? **Answer 9.2:** 10.

This agrees with the pattern. Ten (10) has one zero.

9.2 Zero

Example 9.3: What is 10^0? **Answer 9.3:** 1.

At first, you might think that 10^0 isn't a number, or that it is zero. But, keep in mind the pattern. Ten raised to the zeroeth power should be a number that has zero zeroes. Thus, the pattern so far can be shown as:

$$
\begin{aligned}
10^0 &= 1 \\
10^1 &= 10 \\
10^2 &= 100 \\
10^3 &= 1000
\end{aligned}
$$

and so on.

9.3 Negative integers

Example 9.4: What is 10^{-1}? **Answer 9.4:** 0.1

Again, at first, you might think that this can't be done. However, we can continue to move the decimal point to the left. Thus, $10^{-1} = 0.1$, $10^{-2} = 0.01$, etc. This is also equivalent to $10^{-1} = 1/10$, $10^{-2} = 1/100$, etc.

9.4 Fraction powers of 10

From what was said in the previous section, one can see that ten raised to the power of two equals 100 and ten raised to the power of three equals 1000.

In every situation we discussed, the power was an integer number. What happens if ten is raised to a fractional number, like 10 raised to the power of 2.5 (i.e., $10^{2.5}$)?

Again, you might think that this can't be done, but you should know by now that we are trying to extend the pattern as much as possible. So, let's take the attitude that, if there is an answer, what is it?

Well, first of all, we know that since 2.5 is between 2 and 3, then $10^{2.5}$ should be between 10^2 and 10^3. Thus, it should be between 100 and 1000. This is a very good first estimate.

Now, you might guess that, since 2.5 is exactly between 2 and 3 that $10^{2.5}$ must be exactly between 10^2 and 10^3, which is 550 (the average of 100 and 1000), or perhaps just 500. This turns out to be incorrect.

One can show that it is incorrect by considering whether 10^2 is exactly between 10^1 and 10^3. In other words, is 100 exactly between 10 and 1000?

No. The average of 10 and 1000 is 505. One hundred (10^2) is less than the average of 10^1 and 10^3.

So, we know that $10^{2.5}$ is probably less than 550 (the average of 10^2 and 10^3). But where?

To find out, use a calculator. You should get a number like 316. Most calculators have a y^x button you can use. If you don't, first calculuate 10^5

(i.e., $10 \times 10 \times 10 \times 10 \times 10$) and then take the square root of it. This is the same thing since

$$(10^5)^{1/2} = 10^{5/2} = 10^{2.5}$$

9.5 Logarithm terminology

By now, you should be able to figure out 10^x for any x (estimating the value for non-integer x). Going backwards is a little trickier but similar.

Example 9.5: To what does ten have to be raised to get 100?

Answer 9.5: 2.

The power of 10 that gives the value you want is called the *logarithm* of the value, and is indicated as log(value). Thus, $\log(100) = 2$ and $\log(316) = 2.5$.

Note that normally we wouldn't know that the log of 316 is 2.5. It is only because we did this one before on the calculator that we happen to know it. For values that are not integers, we just have to guess. For example, if we are ask what power ten must be raised to get 500, we could only guess that it has to be between 2.5 and 3.

Fortunately, your calculator can again be used to get the actual value. Your calculator actually has a button that determines the logarithm of any number. Try determining the logarithm of 100, 316 and 500 with your calculator.

9.6 Logarithm of different bases

In the examples shown thus far, everything was a power of 10. Ten, in this case, is called the *base*. We could, however, determine the value that is a power of any number of all, e.g., 2.

Example 9.6: To what does two have to be raised to get 4?

Answer 9.6: 2.

The power of 2 that gives the value you want is also called the logarithm of the value. However, to represent that it is a power of 2, rather than 10, we need to indicate that somehow. For this reason, we usually call it "the logarithm base 2" of the value. Mathematically, this is indicated as $\log_2(\text{value})$.

Example 9.7: What is $\log_2 8$? **Answer 9.7:** 3.

As mentioned above, the base can be any number at all. Sometimes, it is desirable to use e, which has an approximate value of 2.7178 as the base. In this case, rather than writing $\log_e(\text{value})$, we write $\ln(\text{value})$ and call it the *natural log* of the number. Even though we write it differently, it still has all the same properties as any other base.

9.7 Two properties of the logarithm

To simplify problems, it is useful to identify certain properties of the logarithm (or "log" for short) and to take advantage of these properties whenever possible.

To illustrate these properites, we'll suppose we have a value A, which can also be represented as 10^x:

$$10^x = A$$

In this case, ten is raised to the x power so, by definition, x is the logarithm of A:

$$x = \log(A)$$

Suppose, further, that we have another number B, which can be represented as 10^y:

$$\begin{aligned} 10^y &= B \\ y &= \log(B) \end{aligned}$$

9.7.1 $\log B^x = x\log(B)$

The first property of logarithms is used to simplify values that have the form $\log(B^x)$.

Since B can be written as 10^y, we can replace it within the log:

$$\begin{aligned}\log(B^x) &= \log(B^x)\\ &= \log([10^y]^x)\\ &= \log(10^{yx})\end{aligned}$$

which is simply yx. Since $y = \log(B)$, this can also be written as $x\log(B)$.

Putting it all together, we have that

$$\log(B^x) = x\log(B)$$

It is as if we just took the power of B and "dropped" it in front.

✍ You can go through this process every time you encounter a similar problem, but it is probably better just to recognize that this relationship holds.

9.7.2 $\log(AB) = \log(A) + \log(B)$

Now consider $\log(AB)$. Replacing, we get

$$\begin{aligned}\log(AB) &= \log(10^x 10^y)\\ &= \log(10^{x+y})\\ &= x + y\end{aligned}$$

which is simply $\log A + \log B$.

Notice how powers add when the two values are multiplied. Since the logarithm is the power, the logarithm of a product is the sum of the two logarithms.

The same holds true for division.

✍ We can't simplify $\log(A + B)$.

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